Memory driven Ginzburg-Landau model

Steffen Trimper* and Knud Zabrocki
Fachbereich Physik, Martin-Luther-Universität, D-06099 Halle, Germany

Michael Schulz†
Abteilung Theoretische Physik, Universität Ulm, D-89069 Ulm, Germany

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The time evolution of a bistable Ginzburg-Landau model (GL) with a non-Markovian memory term of strength λ is studied. Due to the nonlinear feedback coupling, the two branches of the stationary solution are not only controlled by the sign of the initial condition P_0, but also by the strength and the sign of λ. Whereas in case of a positive λ the stationary solution is ever reduced through the memory, it may be increasing for $\lambda<0$. In that case the system is also able to switch over between both branches of the stationary solution. Such an ability is exclusively achieved for a negative λ within an interval $-u<\lambda<\lambda_c$, where λ_c is a critical memory strength and u is the strength of the conventional nonlinear term within the GL. The complete phase diagram is presented in the P_0-λ plane analytically and numerically.

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I. INTRODUCTION

The crucial factor governing the dynamics of systems comprizing many ''units'' such as people, species, cells, financial transactions, etc., consists of interaction and competition. Such features are believed to underlie the complex dynamics observed in disciplines as diverse as economics [1–4], biology and weather [5], politics [6], medical care [7], and ecology [8]. Another characteristic of physical [9,10] as well as biological systems [11] is time delayed feedback coupling. Let us illustrate it by considering a certain amount of money, available for financial transactions, at present time. Obviously that amount depends on the history of the sample to which it belongs. In the same sense, the change in growth rate of a population should be influenced by the change in growth rate in the past. Moreover, the time evolution of an order parameter could be determined by non-Markovian processes. Thus the evolution equation for a certain quantity, denoted as $P(t)$, has to be supplemented by a memory term. Such a term models, for instance, the way in which a seed capital had been accumulated by rates of interest, the yield, the business on the stock market. In other words, the changing rate of a certain quantity at time t is also determined by the accumulation rate at a former time $t'<t$. In between, i.e., within the interval $\tau=t-t'$, the owner of the capital is in general interested to augment $P(t)$. Regardless of fluctuations, the available capital at time t depends in a decisive manner on the instantaneous gain and loss of money as well as on the changing rate at former times t'. In the same sense, the time evolution of the deviation from an averaged production rate in economy includes both an increasing part, originated by an improved rationalization and reinvestigation, and a saturation due to the common market situation [12]. In case of a phase transition, especially in strongly disordered systems, the present time evolution of an order parameter may be coupled to the time evolution in the past. Hence, it seems to be worthwhile to study simple models, which still conserve the crucial dynamical features of evolution models as nonlinearities and moreover as a new ingredient, delayed feedback-coupling. In the present paper we propose a model, where such retardation effects, characterized by a memory kernel $K(t)$, are taken into account explicitly. Such a memory kernel can be derived following the well established projector formalism proposed in Ref. [13], see also Ref. [14]. Recently, one of us had been successful in finding a nonlinear evolution equation of Fokker-Planck type [15] with memory. The form and the relevance of that term had been discussed by analytical [15] and numerical methods [16]. Notice that the approach had also been very powerful in studying the freezing processes in glasses [17,18]. Our approach yields nontrivial analytical results by assuming that the kernel is originated by the basis quantity $P(t)$ in a self-organized manner, i.e., $K(t)$ is determined by $P(t)$ itself. In detail, we consider the time-dependent-Ginzburg-Landau model (GL), supplemented by a memory term, where that feedbackcoupling is likewise nonlinear in the basis quantity $P(t)$. Consequently, the model offers an additional competition between the nonlinearity, inherent in the GL, and the nonlinear memory term. The final results depend decisively on the strength and the sign of the feedback coupling, denoted by λ, and moreover on the initial conditions P_0. There exists a rich phase diagram in the P_0-λ plane. The analytical results are strongly supported by numerical calculations. Note that the influence of a global feedback has been studied recently in a bistable system [19]. The purpose of that paper was a discussion of the domain-size control by a feedback. However, the approach and the basis equations are completely different from the present paper.

II. THE MODEL

In this section we introduce a model for the time evolution of a quantity $P(t)$, which can be considered as the order parameter, a certain amount of money available at the

*Electronic address: trimper@physik.uni-halle.de
†Electronic address: michael.schulz@physik.uni-ulm.de

present time t, a population, a production, rate, etc. As stressed in the Introduction, the time evolution of $P(t)$ should also be determined by the time evolution in the past. Following the idea presented in Ref. [15], we propose the following non-Markovian evolution equation:

$$\partial_t P(t) = r(P)P(t) - u(P)P(t) - \lambda \int_0^t K[t-t';P(t')]\partial_{t'}P(t')dt'. \qquad (1)$$

The first term characterizes the gain term due to the expenses whereas for generality the input rate $r(P)>0$ can also depend on the instantaneous amount $P(t)$. The case of a constant rate $r(P)=r>0$ (low temperature phase in case of phase transitions) means that r percentage of the available money is obtained as income. Because this case gives rise to an exponential increase of $P(t)$, one has to balance the gain by a loss term $u(P)$. To get a stable gain for positive rates r and u, the loss term is assumed to be $u \propto P^2$, which corresponds to the time-dependent GL. Notice that the generic behavior, discussed below, is not changed by assuming other terms with different power laws in Eq. (1). Obviously, there occurs a competition between the loss and the gain term which yields a stable fixed point. This behavior is strongly modified by a memory term which can be realized by a feedback coupling. To that aim let us assume that the changing rate $\partial_t P(t)$ at the time t is determined by the rate at a previous time $t'<t$. There is a general scheme to derive equations of type (1) [14], where one starts from the microscopical equations. By projecting out all the irrelevant variables, the procedure leads to an equation of type Eq. (1), which is supplemented by a special self-consistent realization of the kernel $K(t)$, the basis equation of our analysis. The parameter λ in Eq. (1) characterizes the strength of the memory. Here we demonstrate that the results are strongly influenced by the sign of λ, provided the kernel is for instance positive definite. If $\lambda<0$, then an accumulation of the capital at the former time t', i.e., $\partial_{t'}P(t')>0$, leads to a further increase at time t whereas the opposite situation, $\lambda>0$, gives rise to a decrease of money at t. A positive definite kernel $K(t)$ is assumed to be valid for glasses [17,18]. Obviously, the projector formalism offers that the kernel is given by the relevant variable $P(t)$ itself. Insofar, the memory effects are self-organized by the system. Thus, it seems to be a reasonable assumption that the time scale of the memory kernel $K(t)$ is determined by the time scale of $P(t)$. Therefore, we suppose $K(t;P(t)) \equiv K(P(t))$, i.e., the memory term is fixed by $P(t)$, accessible in the intermediate time interval $t-t'$. Furthermore, the memory kernel is assumed to be a regular function of $P(t)$. Hence, $K(t)$ can be expanded with respect to $P(t)$, see also Ref. [20]. It results in

$$K(t) = P^{\alpha}(t)\sum_{\nu=0}^{\infty} c_{\nu}P(t)^{\nu}, \qquad (2)$$

where the most interesting case is realized for $\alpha=2$ by neglecting all higher order terms. In that case the memory term may be an additional competitive one compared to the nonlinear term with the prefactor u in Eq. (1). The basis equation for the forthcoming studies now reads

$$\partial_t P(t) = rP(t) - uP^3(t) - \lambda \int_0^t P^2(t-t')\partial_{t'}P(t')dt', \qquad (3)$$

where we suppose $r>0$ and $u>0$. As stressed above, the parameter λ can be positive or negative. Whereas for $\lambda>0$ both nonlinear terms are loss terms, a negative feedback-coupling $\lambda<0$ leads to a competitive situation. The memory kernel gives rise to a coupling between different time scales. In the vicinity of the upper limit of the integral $t' \simeq t$ the memory term reads $K(0)\partial_t P(t)$, i.e., a momentary change at the observation time t is coupled to the value at the initial time $t=0$. Therefore the very past is related to the instantaneous value of $P(t)$. In the opposite case, at the lower limit $t' \simeq 0$, the change of the order parameter near to the initial value $\partial_{t'}P(t'=0)$ is directly coupled to the instantaneous value $K(t)$. Insofar the memory term represents a weighted coupling of time scales, namely between the initial time and the present one. In this manner one should expect that the behavior of the system will be changed due to the memory effects.

III. ANALYTICAL RESULTS

In this section we find the stationary solutions of Eq. (3) and discuss their stability.

A. Stationary solutions

The solution of the evolution Eq. (3) is simple when the memory kernel is zero, i.e., $\lambda=0$. It results in

$$P^2(t) = \frac{P_s^2}{1+(w^2-1)\exp(-\Lambda_0 t)}$$

with

$$P_0 = P(t=0), \quad w = \frac{P_s}{P_0}, \qquad (4)$$

where the nontrivial stationary solution in the long time limit offers the two branches

$$P_s = \pm\sqrt{\frac{r}{u}}. \qquad (5)$$

The quantity w, introduced in Eq. (4), measures the profit in case of $w>1$. A linear stability analysis leads to

$$P(t) \simeq P_s + (P_0 - P_s)\exp(-\Lambda_0 t) \quad \text{with} \quad \Lambda_0 = 2r. \qquad (6)$$

Solution Eq. (4) shows that the sign of the initial value P_0 determines the sign of $P(t)$ uniquely. A positive start capital leads consistently to a positive stationary solution P_s with even $P(t)>0$ for the whole time interval $0 \leqslant t < \infty$. Although the stationary solution is independent of the initial value, the sign of P_0 selects the branch to which P_s belongs. In other words, using a financial interpretation, a positive seed capital can never lead to debts. The inclusion of a memory term will change that behavior drastically. To that aim let us analyze how the profit rate will be controlled by the feedback coupling. For nonzero memory term the solution of Eq. (3) can be found using Laplace transformation, defined by $\mathcal{L}(P(t)) \equiv P(z) = \int_0^\infty P(t)\exp(-zt)dt$. We get

$$P(z) = \frac{P_0 - \dfrac{uB(z)}{1+\lambda A(z)}}{z - \dfrac{r}{1+\lambda A(z)}}$$

with

$$A(z) = \mathcal{L}(P^2(t)) \quad \text{and} \quad B(z) = \mathcal{L}(P^3(t)). \tag{7}$$

A time persistent solution is obtained by making the ansatz $P(t) = f + \Phi(t)$ or after $\mathcal{L}$,

$$P(z) = \frac{f}{z} + \Phi(z), \tag{8}$$

where the function $\Phi(z)$ remains regular for $z \to 0$. The quantity f represents the order parameter in the limit $t \to \infty$. Apart from the trivial solution $f=0$ there exist two new branches of nontrivial solutions,

$$F_\pm \equiv \frac{f_\pm(x,w)}{P_0} = \frac{x}{2(1+x)}\left[1 \pm \mathrm{sgn}(P_0 x)\mathrm{sgn}(1+x) \times \sqrt{1 + \frac{4w^2(1+x)}{x^2}}\right], \quad x = \frac{\lambda}{u}. \tag{9}$$

As remarked before, the parameter λ is assumed to be posititve or negative, whereas u is always positive. Therefore, the dimensionless quantity x can also adopt both signs. In case of vanishing memory it results $f_\pm(x=0,w) = P_s$, independent of P_0 and in accordance with Eq. (5). In general, the solution depends on the initial value P_0. For an infinite memory strength $x \to \infty$ the stationary solutions are given by $F_+ = 1$ and $F_- = 0$, i.e., the parameters r and u are irrelevant. This is consistent with the assumption $r = u = 0$ from the beginning. If one starts with $P_0 = P_s$ it results $F_+ = 1$ or $F_- = -(1+x)^{-1}$.

B. Phase diagram

In this subsection the phase diagram is obtained by employing linear stability analysis. Note that it cannot be performed in terms of $\Phi(z)$ defined in Eq. (8). Instead of that we have to insert $P(t) = f + \Phi(t)$ in Eq. (3). As the result we find $\Phi(t) \propto \exp(-\Lambda t)$, where the stability exponent $\Lambda = f^2(3u+\lambda) - r$ can be rewritten as

$$\frac{\Lambda_\pm}{r} = -1 + \frac{F_\pm^2}{w^2}(3+x) \equiv -1 + \frac{3+x}{1+x}\left(1 + \frac{xF_\pm}{w^2}\right), \quad x \neq -1. \tag{10}$$

The parameter x characterizes the influence of the memory $x = \lambda/u$. The stationary solutions $F_\pm$ are given in Eq. (9). The special case $x=-1$ will be discussed separately. Here we study the stability as well as the stationary solution itself in terms of the free parameter x, where the gain and loss parameters r and u are assumed to be fixed. If the memory strength is positive $\lambda > 0$ or alternatively $x > 0$, then both nonlinear terms in the basis equation (3) with the coupling constants u and λ, respectively, become loss terms, and therefore the memory term should only modify the stationary solution. Indeed, we get in accordance with the general discussion, presented in Sec. II, the following results. If the initial value $P_0 > 0$ and further larger than the stationary value P_s without memory, see Eq. (5), i.e., $w < 1$, then the time delay leads to a reduced gain $P_0 > f_+ > P_s$, whereas in the opposite case $w > 1$, the memory does not reveal an additional enhancement of the capital $P_0 < f_+ < P_s$. Both solutions are stable for all values of the parameter $x > 0$. If one starts with debts, i.e., $P_0 < 0$, the stable solution is f_- and all the results are valid in the same manner as for f_+. The situation is comparable with that one obtained for a zero memory term $x=0$.

The situation becomes more complicated in case of $\lambda < 0$, because the nonlinear terms in Eq. (3) are competitive ones. Moreover, the behavior depends decisively on the ratios $x = \lambda/u < 0$ and on $w = P_s/P_0$, respectively. In discussing the complete solution we have to distinguish three cases, (i) $-u < \lambda < 0$, (ii) $\lambda = -u$, and (iii) $-3u < \lambda < -u$. The previous restriction to $-3 < x < -1$ is originated by the observation that for $x < -3$ the stationary solution is always unstable in accordance with the second part of Eq. (10).

(i) $-1 < x < 0$ or $|\lambda| < u$. In the case of a small memory strength there exist two real solutions $F_\pm$. From Eq. (9) we conclude

$$F_+(P_0 > 0) = F_-(P_0 < 0),$$
$$F_-(P_0 > 0) = F_+(P_0 < 0). \tag{11}$$

Therefore, further discussion can be restricted to $P_0 > 0$, i.e., positive seed capital. For $w > 1$ we get a positive solution with

$$f_+(x,w) > P_s > P_0 \quad \text{or} \quad F_+(x,w) > w > 1. \tag{12}$$

Inserting the last result into Eq. (10), we find that the stability exponent fulfills the relation $\Lambda_+ > 0$, and consequently, the solution is stable within the whole range of the memory parameter $-1 < x < 0$. Thus, a small negative memory strength leads to a real increase of the order parameter $P(t)$. The gain is greater than that obtained without memory effects. In case of $w < 1$ or $P_s < P_0$ there exists a critical value x_c above which the order parameter $P(t)$ can change its sign, i.e., starting with a positive initial value $P_0 > 0$, the system is able to reach the negative stationary solution $f_- < 0$. In other words, the system may switch over to the negative branch of the stationary solution which is impossible without the memory term. To study that problem in detail, we have to analyze the stability exponent Λ. From Eq. (10) it results

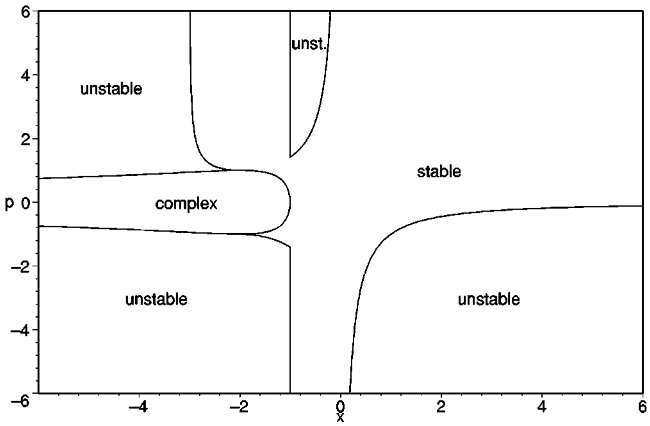

FIG. 1. Phase diagram in the p-x plane with $p=P_0$. The phase boundaries are the zeros of the stability exponent Λ_+ given by Eq. (10).

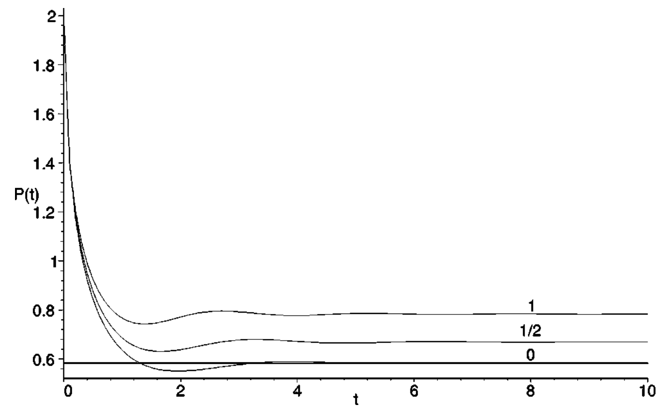

FIG. 2. Temporal evolution of $P(t)$ with $\lambda=-0.8$ and $P_0=2$ for different methods discussed in Sec. IV, parameter $\Theta=0,1/2,1$; step size $\tau=0.1$.

only one stable solution within the interval $1/\sqrt{2}<w<1$ in the whole range $-1<x<0$. However, for $0<w<1/\sqrt{2}$ the exponent Λ_+ is only positive for $x_c<x<0$, where x_c is determined as a solution of $x^2(3+x)=4w^2$. This equation offers three real solutions where only one of them is situated in the interval $-1<x<0$. The critical memory strength follows from the relation

$$x_c=\sqrt{3}\sin(\varphi)-\cos(\varphi) \quad \text{with} \quad \varphi=3\arccos(2w^2)$$

$$w<\frac{1}{\sqrt{2}}. \tag{13}$$

If $w\ll 1$ the critical parameter is $\lambda_c\simeq -2\sqrt{u}P_s/(3|P_0|)$. For $x>x_c$ the solution F_+ becomes unstable and as the consequence, the always stable solution $F_-<0$ is adopted, i.e., $P(t)$ is able to cross the t axis during its time evolution. Insofar, a negative memory strength allows a change over from gain to loss of capital. Due to the symmetry of the problem, manifested by Eq. (11), one can also realize the inverse situation by starting with a negative initial value (debts) and ending up with a positive stationary solution. Such a behavior is driven by the memory effect exclusively. The discussed behavior can be likewise verified, at least quantitatively, using the basis equation (3). Let us assume a positive initial value $P_0>0$ and further suppose the existence of a finite time $t_1>0$ at which $P(t)$ fulfills $P(t_1)=0$. Such a behavior can be realized if $\partial_{t'}P(t')<0$ is fulfilled in the complete time interval $0<t'<t_1$. A further decrease of $P(t)$ is only guaranteed for $\lambda<0$ as it follows from Eq. (3) directly. The results can be further confirmed by a numerical approach, discussed in Sec. IV.

(ii) $x=-1$. If both loss terms are balanced out, the memory parameter is fixed by $|\lambda|=u$. In that case the initial value P_0 has to be as the free parameter. Both stationary solutions are degenerated with $F=w^2$. Furthermore, the sign of the seed capital determines the sign of $P(t)$ in the whole parameter space. The stability exponents reads

$$\frac{\Lambda}{r}=2w^2-1.$$

Therefore we get a stable solution if $P_0<P_c=\sqrt{2}P_s$ whereas in the opposite case the solution is unstable. Note that in this case the initial value P_0 is overrated because the stationary solution leads to $f\propto P_0^{-1}$, which indicates that the model is not very appropriate in that case. On the other hand, the result points to the decisive influence of the time delay effects to the stationary solution of a nonlinear model.

(iii) $-3<x<-1$. In contrast to the other cases both solutions $F_\pm(P_0>0)$ are positive with

$$F_\pm(P_0>0)=\frac{1}{2(1+x)}[x\mp\sqrt{x^2+4w^2(x+1)}]>0. \tag{14}$$

Thus, starting with a positive initial value P_0, the evolution ends up always at a positive stationary value. Due to the symmetry properties given by Eq. (11) we conclude that a negative P_0 never leads to a positive stationary value. Therefore, the discussion can be restricted again to a positive seed capital. If $w>1$, the solutions are real for $x\geqslant x_1=-2w^2+2w\sqrt{w^2-1}$ or $x\leqslant x_2=-2w^2-2w\sqrt{w^2-1}$. Otherwise there appear complex solutions. The stability exponents fulfill the relation

$$\Lambda_\pm<0 \quad \text{for} \quad -3<x<x_3$$

and

$$\Lambda_->0 \quad \text{for} \quad x_3<x<-1, \tag{15}$$

where x_3 is the corresponding solution of the equation $x^2(3+x)=-4w^2(x+1)$. There appears only one stable solution F_- within the interval $x_3<x<-1$. In case of $1/\sqrt{2}<w<1$ we find

$$\Lambda_+>0 \quad \text{for} \quad -3<x_4<x<-2$$

and

$$\Lambda_->0 \quad \text{for} \quad x_5<x<-1, \tag{16}$$

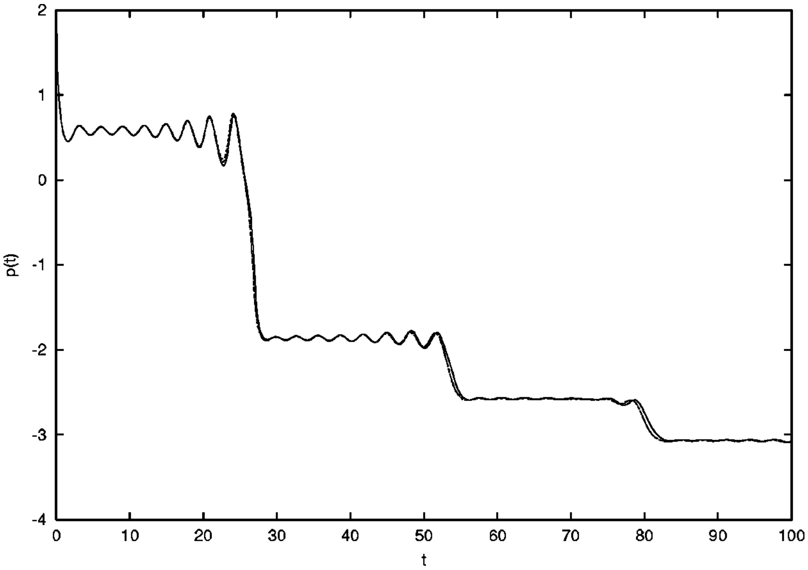

FIG. 3. Temporal evolution of $P(t)$ for negative memory $\lambda=-0.8$ and $P_0=2$ with a high time resolution, step size $\tau=0.001$. The curves for different $\Theta=0,1/2,1$—parameters are degenerated for such a high resolution.

where x_4 and x_5 are the solutions of the equation $x^2(3+x)=4w^2$. If $1<w<1/\sqrt{2}$ the solution F_- is unstable in the whole range of the memory strength $-3<x<-1$, whereas the solution F_+ is stable in the interval $x_4<x<-2$. Let us shortly discuss the special case of a zero initial value $P_0=0$. Here, the stationary solution and the stability exponent read

$$f_\pm=\pm P_s\sqrt{\frac{1}{1+x}},\quad \frac{\Lambda}{r}=\frac{2}{1+x}\quad x\neq-1.$$

For $-1\leqslant x<\infty$ the solution is stable whereas for $x<-1$ the solution is unstable. Summarizing all the different cases we get a phase diagram within the P_0-x plane, which is depicted in Fig. 1. Here, the sign of the stability exponent Λ_+ is represented for fixed parameters r and u. As discussed before, we get stable and unstable solutions, and furthermore a unphysical complex solution discussed before Eq. (14). The phase boundaries are found analytically by calculating the zeros of the stability exponent Λ in Eq. (10). More details of the numerical approach are discussed in the following section.

IV. NUMERICAL APPROACH

To illustrate the analytical results outlined in the preceding section, the application of numerical approximations of the integrodifferential Eq. (3) is efficient. Because of the nonlinearity one cannot expect solutions in a closed form. Instead of that we have applied numerical techniques to figure out the solution $P(t)$. Introducing discrete time steps by $t_n=n\tau$ (τ is the step size) we discuss the evolution equation

$$P_{n+1}=P_n+\tau\left[rP_n-uP_n^3-\lambda\sum_{j=0}^{n}v_j^nP_{n-j}^2(P_{j+1}-P_j)\right].\tag{17}$$

The weights v_j^n depend on the quadrature rule which is applied. Here, we use the so-called general Θ rule discussed for another class of integrodifferential equations of convolution type, see Ref. [21]. In that case the weights are assumed as follows:

$$[v_0^n,v_1^n,\ldots,v_{n-1}^n,v_n^n]=[\Theta,1,\ldots,1,1-\Theta],$$

where $\Theta=0$ leads to the implicit Euler formula, $\Theta=1$ corresponds to the explicit Euler-formula, and $\Theta=1/2$ is the trapezium rule. In our approach we have mainly used the explicit Euler formula to compare the asymptotic behavior of $P(t)$, compare Eq. (9), with the numerical ones. In Fig. 2 one can find the results for different realizations of the parameter Θ, but using the same step size $\tau=0.1$. For such a low resolution of the step width the numerical results suggest that the time evolution of $P(t)$ tends to the unstable fixed point and the solution remains positive. A higher resolution with a reduced step size offers a more pronounced behavior represented in Figs. 3 and 4 in case of $x<0$. The finer the time grid chosen, the smaller is the difference between the different Θ methods, in Fig. 3 the curves for $\Theta=0,1/2,1$ are degenerated. The oscillations, visible in Figs. 3 and 4, are not yet suppressed in a decisive manner. It remains an open problem, whether the oscillations are due to the feedback coupling or induced by the discretization procedure. Our numerical approach, by reducing systematically the time grid, suggests that the oscillations and the steps are an intrinsic effect of the nonlinear system. A possible scenario could be related to the occurence of metastable states, in which the system is confined temporarily. Under the influence of the feedback coupling the system is able to escape from those metatstable states. Equation (3) offers in case of $\lambda<0$ an decreasing derivative of $\partial_t P(t)<0$ if it had been decreased in the past $\partial_{t'}P(t')<0$ with $t'<t$. Both Figs. 3 and 4 illustrate that the positive stationary solution becomes unstable and the trajectory $P(t)$ changes its sign abruptly and crosses the t axis. Likewise in Fig. 4 the time evolution of $P(t)$ is shown for the same negative value of the memory strength,

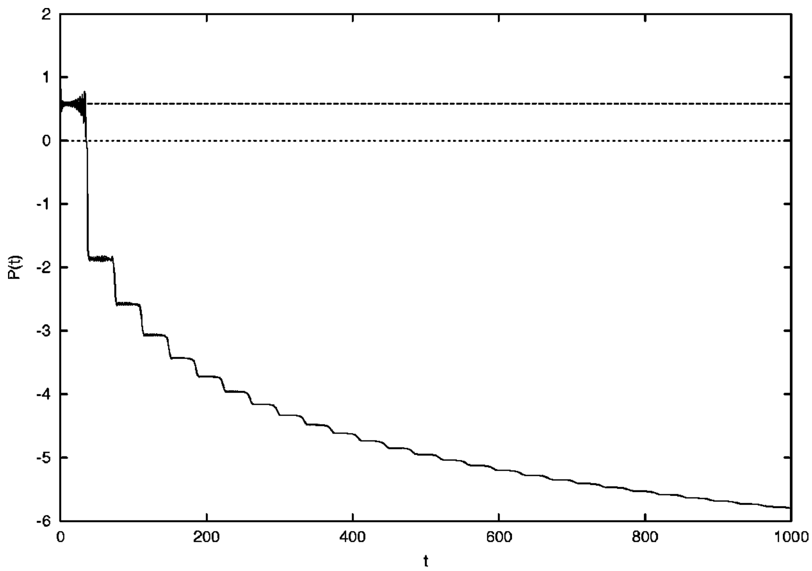

FIG. 4. Temporal evolution of $P(t)$ for $\lambda < 0$ and $\Theta = 0$. The dotted line corresponds to the positive fixed point which becomes unstable. $P(t)$ changes its sign.

but for a longer time scale. Notice that the crossing scenario is in accordance with the analytical results. It corresponds to the unstable island in the upper half of Fig. 1. The effect is traced back to the feedback coupling in a nonambiguous manner.

V. CONCLUSIONS

In this paper we have discussed a more general time-dependent Ginzburg-Landau model by including a self-organized memory coupling, i.e., the instantaneous changing rate of the order parameter is not only determined by time-local gain and loss terms, but additionally by the changing rate in the past. Consequently it means in terms of a financial transaction, the flow of the money is determined by the accumulation or the loss of capital at previous times. To capture the influence of such a delay process, a memory term had been included into the evolution equation. To find out an analytical form of such a feedback coupling, projection methods of statistical mechanics are adopted. Especially, the form of the memory is suggested by investigation in glasses or the anomalous diffusion of particles in a disordered media. We are aware that the form of the feedback term is the most controversial point. However, the time delay effects are introduced in our approach in such a manner that there appears a competitive situation between two nonlinear terms, one is related to an instantaneous loss term whereas the other one is originated by the memory effects. Even that kind of competition leads to a richer behavior as the conventional Ginzburg-Landau model. The main feature of our model consists of a mixing of different time scales, which is manifested by our basis equation (3). The behavior at the current time is related to the rates in the past. Therefore, the system offers the possibility that debts can be reduced by prosperous manipulations in the past although the present situation would indicate an unfavorable development. Another form of the feedback coupling, such as a cumulative one [22], leads to completely different results. In the present paper we have discussed a Ginzburg-Landau-like model with feedback where the stationary solution can be found exactly. Based on a linear stability analysis, we obtain a phase diagram which differs significantly from the standard model. Especially, the feedback reveals the ability of a switchover of the two stationary solutions. The results are supported by numerical solutions. We believe that memory effects are also a feature of other dynamical complex systems.

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